

The cylindrical structure on manifolds via Morse theory

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ABSTRACT. We verify that the cellular stratification(decomposition) on a manifold with a Morse function is cylindrical and show that the associated topological category coincides with the flow category in [CJS].

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1 Introduction

The notion of cellular stratified spaces was introduced in [BGRT] with the aim of constructing a cellular model of the configuration space of a sphere. The theory of cellular stratified spaces was developed by Dai Tamaki in [Tam]. He defined the notion of cylindrically normal cellular stratified spaces and showed that the classifying space $BC(X)$ of the associated topological category $C(X)$ with a cylindrically normal cellular stratified space X can be embedded into X . Moreover, $BC(X)$ is homeomorphic to X if all of the cells are closed.

He said that the notion of cylindrical structure is inspired by the work of Cohen, Jones, and Segal on Morse theory [CJS]. They constructed a topological category C_f for a Morse function $f : M \rightarrow \mathbb{R}$ and showed that the classifying space BC_f is equivalent to M . On the other hand, it is well known that the manifold M has a cellular decomposition with one cell for each critical point of f .

In this paper, we show that the cellular stratification(decomposition) of M is cylindrical and the induced topological category $C(M)$ coincides with C_f . Hence we can show that the result of [CJS] using the method of cellular stratified spaces.

This paper organized as follows. Section 2 describes the definition of cellular stratified spaces and its some properties according to the paper [Tam]. In section 3, we recall the Morse theory for decomposition and reconstruction of manifolds [CJS], [Kal], [Kal'] and show that the decomposition is cylindrically normal.

2 Preliminaries of cellular stratified spaces

Let us recall the definition of cellular stratified spaces. It is a generalization of the notion of cell complexes.

Definition 2.1. Let X be a topological space and Λ be a poset. A stratification of X indexed by Λ is a surjective map $\pi : X \rightarrow \Lambda$ satisfying the following properties:

1. Each $\pi^{-1}(\lambda)$ is connected and locally closed, i.e. it is open in $\overline{\pi^{-1}(\lambda)}$,
2. $\pi^{-1}(\lambda) \subset \overline{\pi^{-1}(\mu)}$ if and only if $\lambda \leq \mu$.

For simplicity, we denote $e_\lambda = \pi^{-1}(\lambda)$ and call it a stratum with index λ . Given a surjective map $\pi : X \rightarrow \Lambda$, we have a decomposition of X , i.e.

1. $X = \bigcup_{\lambda \in \Lambda} e_\lambda$.

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2. For $\lambda, \mu \in \Lambda$, $e_\lambda \cap e_\mu = \phi$ if $\lambda \neq \mu$.

The indexing poset Λ is called the face poset of X and is denoted by $P(X, \Lambda)$ or $P(X)$. We say a stratum e_λ is normal if $e_\mu \subset \overline{e_\lambda}$ whenever $e_\mu \cap \overline{e_\lambda} \neq \phi$. When all strata are normal, the stratification is said to be normal.

Let (X, π_X) and (Y, π_Y) be stratified spaces. A morphism of stratified spaces is a pair (f, \bar{f}) of a continuous map $f : X \rightarrow Y$ and a map of posets $\bar{f} : P(X) \rightarrow P(Y)$ making the following diagram commutative

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ P(X) & \xrightarrow{\bar{f}} & P(Y). \end{array}$$

Definition 2.2. Let X be a topological space. For a non-negative integer n , an n -cell structure on a subspace $e \subset X$ is a pair (D, φ) of a space $\text{Int} D^n \subset D \subset D^n$ and a continuous map $\varphi : D \rightarrow X$ satisfying the following conditions:

1. $\varphi(D) = \bar{e}$.
2. the restriction $\varphi|_{\text{Int}(D)} : \text{Int}(D) \rightarrow e$ is a homeomorphism.
3. the pair (D, φ) is maximal in the poset of pairs satisfying the above conditions for e under inclusions.

We say that an n -cell structure (D, φ) is

- closed if $D = D^n$;
- regular if $\varphi : D \rightarrow \bar{e}$ is a homeomorphism.

A cellular stratification on X is a pair (π, Φ) of a stratification $\pi : X \rightarrow P(X)$ on X and a collection of cell structures

$$\Phi = \{\varphi : D_\lambda \rightarrow \bar{e}_\lambda\}_{\lambda \in P(X)}$$

satisfying the condition that, for each n -cell e_λ , ∂e_λ is covered by a finite number of cells of dimension less than or equal to $n - 1$. A cellular stratified space is a pair $(X, (\pi, \Phi))$ of a space X and a cellular stratification (π, Φ) on X . As usual, we abbreviate it by X , if there is no danger of confusion.

Example 2.3. Let X be a cell complex and e be a n -cell, the characteristic map $\varphi : D^n \rightarrow X$ gives rise to a closed n -cell structure on e . Moreover, the face poset $P(X)$ consists of cells of X whose partial order $e_\lambda < e_\mu$ is given by $e_\lambda \subset \bar{e}_\mu$. Thus the cellular decomposition of X gives a closed cellular stratification on X .

It is well known that the order complex of the face poset of a regular cell complex X is homeomorphic to X . We may also define the face poset of a cellular stratified space. One of the main results in [BGR] is that the order complex of the face poset of a regular totally normal cellular stratified space X can be embedded in X as a strong deformation retract. For non-regular cellular stratified spaces, however, we cannot expect to recover the homotopy type of the original space from its face poset. It suggests that the poset does not have enough information to recover the homotopy type. Hence we construct a topological category for a general cellular stratified space.

Definition 2.4. Let X be a cellular stratified space. The face category $F(X)$ is defined by follows:

- The set of objects is $P(X)$.

- The hom space

$$F(X)(\lambda, \mu) = \{f \in \text{Map}(D_\lambda, D_\mu) \mid \varphi_\lambda = \varphi_\mu \circ f\}$$

as a subspace of $\text{Map}(D_\lambda, D_\mu)$ with the compact open topology. The composition is given by the composition of maps.

Unfortunately, $F(X)$ is too large to reconstruct the original space X . In [Tam], Tamaki considered a subcategory $C(X)$ of $F(X)$ for cellular stratified spaces satisfying some nice properties.

Definition 2.5. A cylindrical structure on a normal cellular stratified space X consists of

- a normal stratification on S^{n-1} containing ∂D_λ as a stratified subspace for each n -cell $\varphi_\lambda : D_\lambda \longrightarrow \overline{e_\lambda}$,
- a stratified space $P_{\lambda, \mu}$ and a morphism of stratified spaces

$$b_{\lambda, \mu} : P_{\lambda, \mu} \times D_\lambda \longrightarrow \partial D_\mu$$

for $\lambda < \mu$, $\lambda \neq \mu$

- a morphism of stratified spaces

$$c_{\lambda_1, \lambda_2, \lambda_3} : P_{\lambda_2, \lambda_3} \times P_{\lambda_1, \lambda_2} \longrightarrow P_{\lambda_1, \lambda_3}$$

for a sequence $\lambda_1 < \lambda_2 < \lambda_3$

satisfying the following conditions:

1. The restriction of $b_{\lambda, \mu}$ to $P_{\lambda, \mu} \times \text{Int}(D_\lambda)$ is a homeomorphism onto its image.
2. The following diagram is commutative

$$\begin{array}{ccc} P_{\lambda, \mu} \times D_\lambda & \xrightarrow{\text{pr}_2} & D_\lambda \\ b_{\lambda, \mu} \downarrow & & \downarrow \varphi_\lambda \\ D_\mu & \xrightarrow{\varphi_\mu} & X. \end{array}$$

3. The following diagram is commutative

$$\begin{array}{ccc} P_{\lambda_2, \lambda_3} \times P_{\lambda_1, \lambda_2} \times D_{\lambda_1} & \xrightarrow{1 \times b_{\lambda_1, \lambda_2}} & P_{\lambda_2, \lambda_3} \times D_{\lambda_2} \\ c_{\lambda_1, \lambda_2, \lambda_3} \times 1 \downarrow & & \downarrow b_{\lambda_2, \lambda_3} \\ P_{\lambda_1, \lambda_3} \times D_{\lambda_1} & \xrightarrow{b_{\lambda_1, \lambda_3}} & D_{\lambda_3}. \end{array}$$

4. The composition map $c_{\lambda_1, \lambda_2, \lambda_3}$ satisfies the associativity condition.
5. We have

$$\partial D_\mu = \bigcup_{\lambda < \mu, \lambda \neq \mu} b_{\lambda, \mu}(P_{\lambda, \mu} \times \text{Int}(D_\lambda))$$

as a stratified space.

The space $P_{\lambda, \mu}$ is the parameter space for $\lambda < \mu$, where $P_{\lambda, \mu} = *$ if $\lambda = \mu$. A normal cellular stratified space equipped with a cylindrical structure is called a cylindrically normal cellular stratified space.

Definition 2.6. Let X be a cylindrically normal cellular stratified space. The cylindrical face category $C(X)$ is defined by follows:

- The set of objects is $P(X)$.
- The hom space $C(X)(\lambda, \mu) = P_{\lambda, \mu}$. The composition is given by $c_{\lambda_1, \lambda_2, \lambda_3}$.

The map $b_{\lambda, \mu} : P_{\lambda, \mu} \times D_\lambda \rightarrow D_\mu$ induce $P_{\lambda, \mu} \rightarrow F(X)(\lambda, \mu) \subset \text{Map}(D_\lambda, D_\mu)$. It gives a continuous functor $b : C(X) \rightarrow F(X)$ over the face poset $P(X)$.

The following is a main result in [Tam].

Theorem 2.7 (Tam). *Let X be a cylindrically normal closed cellular stratified space, then the classifying space $BC(X)$ is homeomorphic to X .*

3 The cylindrical structure on a manifold with a Morse function

The face category $F(X)$ is an acyclic topological category, i.e. $F(\lambda, \mu) = \phi$ if $\mu < \lambda$, $\mu \neq \lambda$ and $F(\lambda, \lambda)$ consists of only the identity morphism. Obviously, $C(X)$ is so. On the other hands, Cohen, Jones and Segal constructed an acyclic topological category C_f for a Morse-Smale function $f : M \rightarrow \mathbb{R}$ such that the classifying space BC_f is equivalent to M . Now we recall the definition of C_f and the their result.

The following notation will be used throughout this paper.

- M is a closed manifold of dimension m .
- f is a Morse function on M with critical points p_1, p_2, \dots, p_n and $f(p_1) < f(p_2) < \dots < f(p_n)$.
- λ_i is the index of f at p_i .
- Δf is the gradient vector field of f .
- $W^s(p_i)$ (resp. $W^u(p_i)$) is the stable (resp. unstable) manifold associated with p_i .
- $M^a = \{x \in M \mid p(x) \leq a\}$.

It is possible to choose a Riemannian metric on M such that for each critical point p_i , there exists a neighborhood U_i in M with coordinate functions x in U_i such that $f(x)$ is given in U_i by

$$f(x) = -x_1^2 - \dots - x_{\lambda_i}^2 + x_{\lambda_i+1}^2 + \dots + x_m^2 + f(p_i).$$

We call f is a Morse-Smale function if $W^s(p_i)$ and $W^u(p_j)$ are transversal for any critical points p_i and p_j . For critical points p and q , let $M(p, q)$ be the moduli space of flow lines from p to q .

Definition 3.1. Let $f : M \rightarrow \mathbb{R}$ be a Morse function. Define the category C_f as follows.

- The set of objects consists of the critical points of f .
- The hom-space $C_f(p, q) = \overline{M}(p, q)$ is the compactification of the moduli space. The composition is given by the composition of piecewise flow lines.

For a Morse-Smale function $f : M \rightarrow \mathbb{R}$, G.Kalmbach [Kal] gives a cell decomposition $M = \bigcup_{1 \leq i \leq n} W^u(p_i)$. We call it the Morse theoretic decomposition of M associated with f . The characteristic map $\varphi : D^{\lambda_i} \rightarrow M$ is given by the following. We can regard D^{λ_i} as

$$\left\{ (x_1, \dots, x_m) \mid \sum_{k=1}^{\lambda_i} x_k^2 \leq 2\varepsilon, x_\ell = 0, j > \lambda_i \right\} \subset U_i$$

for a small $\varepsilon > 0$ enough. Define $\varphi(x) = x$ if $0 \leq |x| \leq \varepsilon$ and $\varphi(x) = \delta_{g(x)}(x)$ if $\varepsilon \leq |x| \leq 2\varepsilon$, where $\delta_t(x)$ is the flow line via x of the vector field X for $t \in [0, 1]$ and $g(x) \in [0, 1]$ is a parameter satisfying $g(x) = 1$ if $|x| = 2\varepsilon$ and $\delta_{g(x)}(x) = x$ if $|x| = \varepsilon$. The above vector field X is given in [Kal].

Define

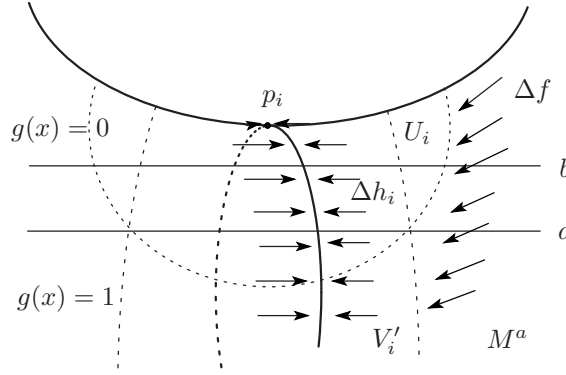
$$\tilde{h}_i(x) = \sum_{i+1 \leq j \leq m} x_{\lambda_j}^2$$

for $x \in U_i$. Let $V_i = \{x \in U_i \mid \tilde{h}_i(x) < \varepsilon\}$ and $V'_i = \{\gamma_t(x) \in M^{f(x)} \mid x \in V_i, t \in \mathbb{R}\}$ where $\gamma_t(x)$ is the flow line via x of Δf . Take $a < b < f(p_i)$ with $\partial M^a \cap V'_i \subset U_i$ and take a separation-function $g = g_{a,b,f}$ satisfying $g(x) = 0$ if $x \in U_i - \text{Int}(M^b)$ and $g(x) = 1$ if $x \in M^a$. Define $\hat{h}_i(x) = \tilde{h}(\gamma_t(x)) \in \partial M^a$ for $x \in V'_i - \{p_i\}$ and

$$h_i(x) = (1 - g(x))\tilde{h}(x) + g(p)\hat{h}(x)$$

for $x \in V'_i$. $\Delta h_i(x) = 0$ if $x \in W^u(p_i)$, otherwise $\Delta h_i \neq 0$.

The vector field X is Δh_i near around $W^u(p_i)$ and X is Δf on faraway places from $W^u(p_i)$.



Lemma 3.2. *The Morse theoretic decomposition of M associated with f is normal.*

Proof. Suppose $W^u(p_i) \cap \overline{W^u(p_j)} \neq \emptyset$. Let $x \in W^u(p_i) \cap \overline{W^u(p_j)}$, there exists a flow line δ of X from p_j to x . Also there is a flow line γ of Δf from p_j to p_i such that δ flows from the γ -direction. It means that δ is a flow line satisfying the following. Take a small $s > 0$ and a big $M > 0$ enough. For $x = \beta_t \in W^u(p_i) \cong \text{Int}(D^{\lambda_i})$ where β is a flow line from p_i , $\beta_s = (x_1, \dots, x_{\lambda_i}, 0, \dots, 0) \in U_i$. Similarly, $\gamma_M = (0, \dots, 0, x_{\lambda_i+1}, \dots, x_m) \in W^s(p_i) \cap U_i$. The flow line δ is a unique flow line via $\alpha(x_1, \dots, x_{\lambda_i}, x_{\lambda_i+1}, \dots, x_m)$ where α is a homeomorphism from a neighborhood of γ_s to a neighborhood of γ_t such that $\gamma_s \mapsto \gamma_t$. For $y \in W^u(p_i)$, there exists a flow line of X flowing the γ -direction to y , hence $y \in \overline{W^u(p_j)}$. \square

Theorem 3.3. *The Morse theoretic decomposition of M associated with f is cylindrical and $C(M) = C_f$.*

Proof. For a pair of critical points (p_i, p_j) satisfying $f(p_j) < f(p_i)$, let $P_{p_i, p_j} = \overline{M}(p_j, p_i)$ and regard as a trivial stratified space. Define the map

$$b_{p_i, p_j} : \overline{M}(p_j, p_i) \times D^{\lambda_i} \longrightarrow \partial D^{\lambda_j}$$

as follows. For $(\gamma, x) \in \overline{M}(p_j, p_i) \times D^{\lambda_i}$, there exists a unique flow line δ of the vector field X flowing from the γ -direction to $\varphi(x) \in W^u(p_i)$. Define

$$b_{p_i, p_j}(\gamma, x) \in \partial D^{\lambda_j} \cong \left\{ (x_1, \dots, x_m) \mid \sum_{k=1}^{\lambda_j} x_k^2 = 2\varepsilon, x_\ell = 0, \ell > \lambda_j \right\} \subset U_j$$

by the point that δ passes. Let us verify that b_{p_i, p_j} satisfies the condition of cylindrical structure. Since the restriction of characteristic map $\varphi_{\text{Int}(D^{\lambda})}$ is a homeomorphism onto its

image and the universality of flow lines, the restriction of b_{p_i, p_j} to $\overline{M}(p_j, p_i) \times \text{Int}(D_{\lambda_i})$ is a homeomorphism onto its image. The following diagram

$$\begin{array}{ccc} \overline{M}_{p_j, p_i} \times D^{\lambda_i} & \xrightarrow{\text{pr}_2} & D^{\lambda_i} \\ b_{p_i, p_j} \downarrow & & \downarrow \varphi_i \\ D^{\lambda_j} & \xrightarrow{\varphi_j} & M \end{array}$$

is commutative by the definition of b_{p_i, p_j} . The composition of piecewise flow lines satisfies the associativity condition. For $(\gamma_2, \gamma_1, x) \in \overline{M}(p_k, p_j) \times \overline{M}(p_j, p_i) \times D^{\lambda_i}$, $b_{p_i, p_j}(\gamma_1, x)$ and $b_{p_i, p_k}(\gamma_2 \circ \gamma_1, x)$ belong to the common flow line δ since γ_1 and $\gamma_2 \circ \gamma_1$ flow from the same direction to p_i in U_i . The flow line δ passes through $b_{p_i, p_j}(\gamma_1, x)$ from the γ_2 -direction, then

$$b_{p_k, p_j}(\gamma_2, b_{p_i, p_j}(\gamma_1, x)) = b_{p_i, p_k}(\gamma_2 \circ \gamma_1, x)$$

i.e. the following diagram is commutative

$$\begin{array}{ccc} \overline{M}(p_k, p_j) \times \overline{M}(p_j, p_i) \times D^{\lambda_i} & \xrightarrow{1 \times b_{p_i, p_j}} & \overline{M}(p_k, p_j) \times D^{\lambda_j} \\ \circ \times 1 \downarrow & & \downarrow b_{p_j, p_k} \\ \overline{M}(p_k, p_i) \times D^{\lambda_i} & \xrightarrow{b_{\lambda_1, \lambda_3}} & D^{\lambda_k}. \end{array}$$

The definition of b_{p_i, p_j} implies

$$\partial D^{\lambda_j} = \bigcup_{i < j} b_{p_i, p_j}(\overline{M}(p_j, p_i) \times \text{Int}(D^{\lambda_i}))$$

and it gives a stratification of ∂D^{λ_j} . □

The following theorem is shown by Cohen, Jones and Segal in [CJS]. However, we can show it using the cylindrically normal cellular stratification of M associated with f and Theorem 2.7.

Theorem 3.4. *Let $f : M \rightarrow \mathbb{R}$ be a Morse-Smale function on a closed manifold, then $BC(M) \cong M$.*

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